

Hamiltonian-connectivity and strongly Hamiltonian-laceability of folded hypercubes[☆]

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Abstract

In this paper, we analyze a hypercube-like structure, called the folded hypercube, which is basically a standard hypercube with some extra links established between its nodes. We first show that the n -dimensional folded hypercube is bipartite when n is odd. We also show that the n -dimensional folded hypercube is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or $n (\geq 2)$ is even.

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1. Introduction

Design of interconnection networks is an important integral part of parallel processing and distributed systems. There are a large number of topological choices for interconnection networks (the interested readers may refer to [1–3] for extensive references). Among them, the hypercube [4] has several excellent properties such as recursive structure, regularity, symmetry, small diameter, relatively short mean internode distance, low degree, and very small link complexity, which are very important for designing massively parallel or distributed systems [5]. Since its introduction, many variants of the hypercube have been proposed [6–8]. One variant that has been the focus of a great deal of research is the *folded hypercube*, which is an extension of the hypercube, constructed by adding a link to every pair of nodes that are the farthest apart, i.e., two nodes with complementary addresses. The folded hypercube has been shown to be able to improve the system's performance over a regular hypercube in many measurements [6,9].

$G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) | (u, v) \text{ is an unordered pair of } V\}$. We say that V is the *node set* and E is the *edge set*. We also use $V(G)$ and $E(G)$ to denote the node set and edge set of G , respectively. In this paper, we use graph and network, node and vertex, link and edge, interchangeably. Usually when the Hamiltonicity of a graph G is concerned, it is investigated whether G is Hamiltonian or Hamiltonian-connected. A cycle (resp., path) in G is called a *Hamiltonian cycle* (resp., *Hamiltonian path*) if it contains every node of G exactly

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once. G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if there exists a Hamiltonian path between every two nodes of G .

A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y) | x \in V_0 \text{ and } y \in V_1\}$. We call V_0 and V_1 *partite sets* of G , and $V_0 \cup V_1$ a *bipartition*. Hypercubes [5] and star graphs [10] are both bipartite. However, since a bipartite graph is not Hamiltonian-connected except for K_2 or K_1 , Simmons [11] introduced the concept of Hamiltonian-laceability for those Hamiltonian bipartite graphs. A Hamiltonian bipartite graph $G = (V_0 \cup V_1, E)$ is *Hamiltonian-laceable* if there is a Hamiltonian path between any two nodes x and y , where $x \in V_0$ and $y \in V_1$. Hsieh et al. [12] further extended this concept and proposed the concept of *strongly Hamiltonian-laceability*. A Hamiltonian-laceable graph $G = (V_0 \cup V_1, E)$ is *strongly* Hamiltonian-laceable if there is a simple path of length $|V_0| + |V_1| - 2$ between any two nodes of the same partite set. Simmons [11] showed that the n -dimensional hypercube is Hamiltonian-laceable. Tsai et al. [13] further showed that the n -dimensional hypercube is strongly Hamiltonian-laceable. Hsieh et al. [12] showed that the n -dimensional star graph is strongly Hamiltonian-laceable. In this paper, we first show that the n -dimensional folded hypercube is bipartite when n is odd. We also show that the n -dimensional folded hypercube is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or $n (\geq 2)$ is even.

2. Preliminaries

An n -dimensional hypercube (n -cube for short) can be represented as an undirected graph $Q_n = (V, E)$ such that V consists of 2^n nodes which are labeled as binary numbers of length n from $\underbrace{00 \dots 0}_n$ to $\underbrace{11 \dots 1}_n$. E is the set of edges

that connects two nodes if and only if they differ in exactly one bit of their labels. Thus, each node has immediate links with exactly n other nodes. It can easily be shown that $|E| = n2^{n-1}$. A link (or edge) $e = (v_i, v_j) \in E$ represents the two nodes v_i and v_j which are linked by e and have exactly one bit different. Therefore e can be denoted using the two nodes that it links: If $v_i = b_n b_{n-1} \dots b_k \dots b_1$, $v_j = b_n b_{n-1} \dots \bar{b}_k \dots b_1$ (where $b_l \in \{0, 1\}, l = 1, \dots, n$), and $e = (v_i, v_j)$, then we denote e as $b_n \dots b_{k+1} x b_{k-1} \dots b_1$. We call $b_n \dots b_{k+1} x b_{k-1} \dots b_1$ a link of dimension k . There are 2^{n-1} links in each dimension.

Let $x = x_n x_{n-1} \dots x_1$ be an n -bit binary string. For $1 \leq k \leq n$, we use $x^{(k)}$ to denote the binary strings $y_n y_{n-1} \dots y_1$ such that $y_k = 1 - x_k$ and $x_i = y_i$ for all $i \neq k$. The *Hamming weight* $hw(x)$ of x is the number of i 's such that $x_i = 1$. Let $x = x_n x_{n-1} \dots x_1$ and $y = y_n y_{n-1} \dots y_1$ be two n -bit binary strings. The *Hamming distance* $h(x, y)$ between two nodes x and y is the number of different bits in the corresponding strings of both nodes. Note that an n -cube Q_n is a bipartite graph with bipartition $\{x | hw(x) \text{ is odd}\}$ and $\{x | hw(x) \text{ is even}\}$. Let $d_{Q_n}(x, y)$ be the distance of a shortest path between two vertices x and y in graph Q_n . Then, it is known that $d_{Q_n}(x, y) = h(x, y)$.

An n -dimensional folded hypercube (*folded n -cube* for short) FQ_n is a regular n -dimensional hypercube augmented by adding more links among its nodes. More specifically, a folded n -cube is obtained by adding a link between two nodes whose addresses are complementary to each other; i.e., for a node whose address is $b = b_1 b_2 \dots b_n$, it now has one more link to node $\bar{b} = \bar{b}_1 \bar{b}_2 \dots \bar{b}_n$, in addition to its original n links. So a folded n -cube has 2^{n-1} more links than a regular n -cube. We call these augmented links *skips*, to distinguish them from regular links, and use S to denote the set of skips. So the complete link set $E(FQ_n)$ of a folded hypercube can be expressed as $I \cup S$. In other words, we can formally define the edges of a folded n -cube by $E(FQ_n) = I \cup S = \{e = (u, v) | h(u, v) = 1 \in I \text{ or } h(u, v) = n \in S\}$. Fig. 1 illustrates a two-dimensional and a three-dimensional folded hypercube.

For convenience, a folded n -cube FQ_n can be represented with $\underbrace{** \dots *}_n = *^n$, where $*$ $\in \{0, 1\}$ means the “don't care” symbol. Moreover, $Q_{n-1}^{0i} = *^{n-i} 0 *^{i-1}$ and $Q_{n-1}^{1i} = *^{n-i} 1 *^{i-1}$, which contain the nodes with the i th bits 0 and 1, respectively, represent two node-disjoint $(n-1)$ -cubes. Formally, Q_{n-1}^{0i} (resp., Q_{n-1}^{1i}) is the subgraph of FQ_n induced by $\{x_n \dots x_i \dots x_1 \in V(FQ_n) | x_i = 0\}$ (resp., $\{x_n \dots x_i \dots x_1 \in V(FQ_n) | x_i = 1\}$). Clearly, each Q_{n-1}^{ji} , $j \in \{0, 1\}$, is isomorphic to Q_{n-1} .

Definition 1. An i -partition on $FQ_n = *^n$, where $1 \leq i \leq n$, is a partition of FQ_n along dimension i into two $(n-1)$ -cubes $*^{n-i} 0 *^{i-1}$ and $*^{n-i} 1 *^{i-1}$.

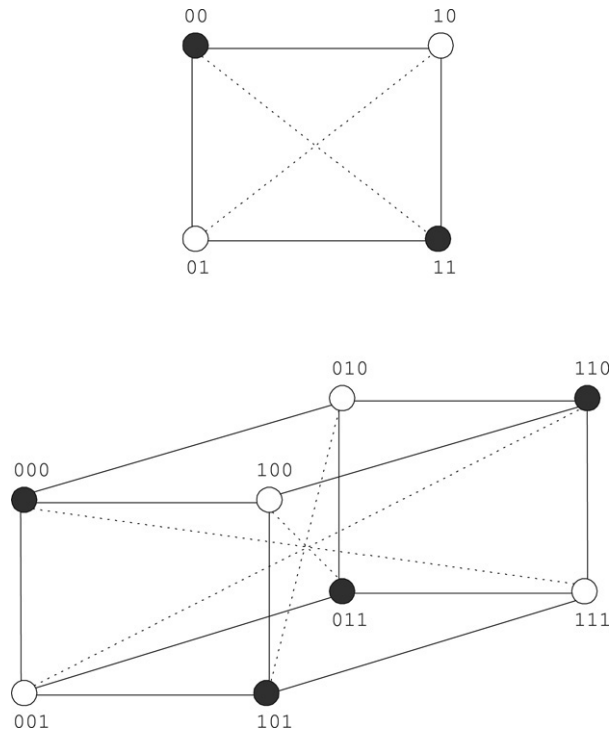


Fig. 1. A folded 2-cube and a folded 3-cube, in which links in skip set S are represented with dashed lines.

In the rest of the paper, a path from x to y is abbreviated as an x – y path. The following lemma was shown in [13], and it will be used in the following section.

Lemma 1 ([13]). *The n -dimensional hypercube is strongly Hamiltonian-laceable. That is, there is a Hamiltonian path with length $2^n - 1$ between two arbitrary nodes in different partite sets, and a longest path with length $2^n - 2$ between two arbitrary nodes of the same partite set.*

3. Main result

In this paper, we show that the n -dimensional folded hypercube is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or $n (\geq 2)$ is even.

Lemma 2. *The n -dimensional folded hypercube FQ_n is both bipartite and strongly Hamiltonian-laceable when n is odd.*

Proof. Clearly, FQ_1 is bipartite and strongly Hamiltonian-laceable. We next show that FQ_n is bipartite when $n (\geq 3)$ is odd. Consider a partition of the node set $V(FQ_n)$ into two sets $X = \{v \in V(FQ_n) | hw(v) \text{ is odd}\}$ and $Y = \{v \in V(FQ_n) | hw(v) \text{ is even}\}$. We next show that if n is odd, then X and Y are two partite sets of FQ_n . Recall that $E(FQ_n) = I \cup S = \{e = (u, v) | h(u, v) = 1 \in I \text{ or } h(u, v) = n \in S\}$.

CASE 1. $(x, y) \in I$. By definition, $h(x, y) = 1$. Thus one endpoint is in X and the other is in Y .

CASE 2. $(x, y) \in S$. Without loss of generality, assume that $hw(x)$ is odd (the case for $hw(x)$ being even can be shown similarly). Since n is odd, and x and y are complementary to each other, we have that $hw(y) = n - hw(x)$ which is even. Therefore, $x \in X$ and $y \in Y$.

By cases 1 and 2, X and Y are independent sets such that all edges of $E(FQ_n)$ are between X and Y . Thus FQ_n is bipartite when n is odd. Moreover, since FQ_n has the same bipartition as that of the n -dimensional hypercube Q_n , and it contains a subgraph isomorphic to Q_n , FQ_n is strongly Hamiltonian-laceable by Lemma 1. \square

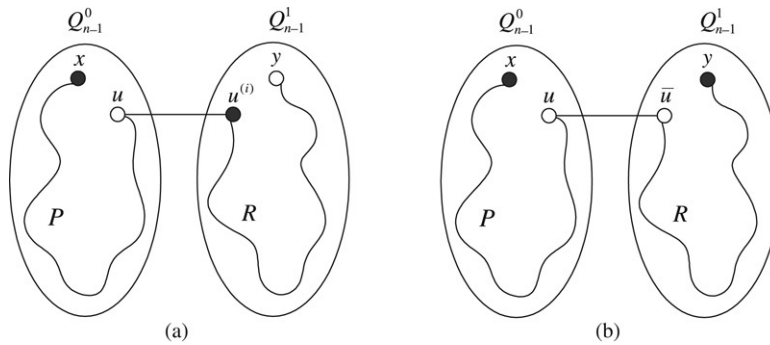


Fig. 2. An illustration of the proof of Lemma 3.

The following lemma shows that a folded n -cube is Hamiltonian-connected, when $n = 1$ or $n (\geq 2)$ is even.

Lemma 3. *The n -dimensional folded hypercube FQ_n is Hamiltonian-connected, when $n = 1$ or $n (\geq 2)$ is even.*

Proof. We show this lemma by induction on the dimension n . Clearly, FQ_1 is exactly an edge and FQ_2 is a complete graph on four vertices. Thus the result holds trivially on the base cases.

We now consider a folded n -cube FQ_n , where $n (\geq 4)$ is even. Let $x = x_n x_{n-1} \cdots x_1$ and $y = y_n y_{n-1} \cdots y_1$ be two arbitrary distinct vertices of FQ_n . Clearly, there exists an i dimension such that $x_i \neq y_i$. After executing an i -partition on FQ_n , two $(n-1)$ -dimensional hypercubes Q_{n-1}^{0i} and Q_{n-1}^{1i} result such that x and y are in the different subcubes. Without loss of generality, assume that $x \in Q_{n-1}^{0i}$ and $y \in Q_{n-1}^{1i}$. Consider the following two cases:

CASE 1. x and y are in the different partite sets (see also Fig. 2(a)). Let u be a neighbor of x in Q_{n-1}^{0i} and u be not adjacent to y . Thus x and u are in different partite sets in Q_{n-1}^{0i} . By Lemma 1, there is a Hamiltonian x, u -path P in Q_{n-1}^{0i} , and P is of length $2^{n-1} - 1$. Since $u^{(i)} \in Q_{n-1}^{1i}$ is adjacent to u , $u^{(i)}$ and y are in different partite sets in Q_{n-1}^{1i} . By Lemma 1, there is a Hamiltonian $u^{(i)}, y$ -path R in Q_{n-1}^{1i} , and R is of length $2^{n-1} - 1$. Therefore, $P + (u, u^{(i)}) + R$ is a Hamiltonian x, y -path of FQ_n .

CASE 2. x and y are in the same partite set (see also Fig. 2(b)). Let u be a neighbor of x in Q_{n-1}^{0i} and y not be the complement of u . Thus x and u are in different partite sets in Q_{n-1}^{0i} . By Lemma 1, there is a Hamiltonian x, u -path P in Q_{n-1}^{0i} , and P is of length $2^{n-1} - 1$. Moreover, \bar{u} is the complement of u in Q_{n-1}^{1i} . Note that $(u, \bar{u}) \in S$ is an edge of FQ_n . Moreover, since n is even, if $hw(u)$ is even (resp., odd), then $hw(\bar{u})$ is even (resp., odd), i.e., $hw(u)$ and $hw(\bar{u})$ have the same parity. Since the parity of x is the same as that of y and is different from that of u, \bar{u} and y are in different partite sets in Q_{n-1}^{1i} . By Lemma 1, there is a Hamiltonian \bar{u}, y -path R in Q_{n-1}^{1i} . Therefore, $P + (u, \bar{u}) + R$ is a Hamiltonian x, y -path of FQ_n .

The proof is completed. \square

Theorem 1. *The n -dimensional folded hypercube FQ_n is Hamiltonian-connected when $n = 1$ or $n \geq 2$ is even; and strongly Hamiltonian-laceable when n is odd.*

4. Concluding remarks

In this paper, we show that the n -dimensional folded hypercube is bipartite when n is odd. Moreover, we also show that it is strongly Hamiltonian-laceable when n is odd, and is Hamiltonian-connected when $n = 1$ or $n (\geq 2)$ is even.

Paths (linear arrays) are the most fundamental networks for parallel and distributed computation, which are suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on paths for solving various algebraic problems and graph problems can be found in [14,5]. They can also be used as control/data flow structure for distributed computation in arbitrary networks. Another application for Hamiltonian paths to a practical problem was addressed in the on-line optimization of a complex flexible manufacturing system [15] and the wormhole routing [16,17]. These applications motivate the embedding of Hamiltonian paths in networks. Our

result implies that those algorithms designed for Hamiltonian paths can also be executed well on the folded hypercube with faulty edges.

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